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Nuclear Physics B17 (1970) 88-116. North-Holland Publ. Comp., Amsterdam

THE THREE-REGGEON VERTEX: ANALYTICITY, ASYMPTOTICS AND THE TOLLER POLE MODEL

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Received 1 October 1969

Abstract: The incorporation of the local analyticity properties of the six-particle amplitude in a previous treatment of the three-Reggeon vertex is considered. A triple Regge pole contribution is shown to be singular on the boundary between two parts of the physical region where the partial-wave expansion was shown to take different forms. This singularity can be removed by appropriate behaviour of the residue function, but the asymptotic region where the pole contribution can be expected to dominate behaves unsatisfactorily. For comparison, the connection between the singularity of a Regge pole contribution and the bad behaviour of the asymptotic region is also discussed for the zero momentum transfer problem. Analytic group variables, uniformly related to the invariants are introduced for the six-particle amplitude. A Lorentz-group expansion incorporating the vertex covariance condition is given and a triple Toller pole shown to be a possible uniform asymptotic approximation to the amplitude in the neighbourhood of the boundary considered.

The treatment of the three-Reggeon vertex is used to give a full group theoretic treatment of an arbitrary multiparticle amplitude.

1. INTRODUCTION

In a previous paper [1] we considered the group theoretic description of the six-particle amplitude corresponding to the "tree-diagram" of fig. 1, which involves a three-Reggeon vertex, without considering analyticity. The three-Reggeon vertex plays a central role in the general formulation of the group theoretic description of an arbitrary multiparticle amplitude, as we discuss later on. Consequently, in this paper we study the incorporation of the local analyticity properties of the amplitude in our approach, and the relation of analyticity problems to the asymptotics of Regge and Toller pole contributions.

Before [1], we discussed in detail the generalised partial wave expansion corresponding to fig. 1 in a physical region where the momentum transfers Q_A, Q_B and Q_C are spacelike. It was necessary to distinguish between two parts of the physical region. In one part, the s-t region, the plane defined by Q_A, Q_B and Q_C contains some time-like vectors and the expansion is very similar to the Bali, Chew and Pignotti expansions [2]. In the second part, the s-s region, this plane contains only space-like

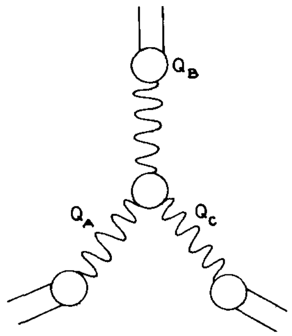

Fig. 1. A three-Reggeon vertex for the six-particle amplitude.

vectors and the expansion takes a different form in which it is effectively necessary to use a continuous basis for the representation functions. At the boundary between these two regions (which occurs inside the physical region), the relation of the group theoretical variables that we used to the invariants is singular. In this paper we show that as a result, a triple Regge pole contribution, from the s-t region, say, will be singular at this point unless the residue has an appropriate branch point. (The behaviour of a triple Regge pole contribution near this boundary is important because it involves a continuous range of individual trajectories). If the residue has this branch point it follows that the contribution will either vanish or retain only its leading behaviour there.

The daughter problem for the four-particle amplitude at $W = 0$ can also be regarded as resulting from the singular relation of the group variables to invariants. This singularity (as we show in sect. 2) results in the asymptotic region in which Regge pole contributions can be expected to dominate receding to infinity. At the boundary between our two regions the asymptotics are also peculiar, but in the inverse way, that is the asymptotic region comes in to finite values of the invariants, and so at $W=0$, we are led to introduce analytic little group variables constructed inside the Lorentz group. These variables, as well as facilitating a precise discussions of analyticity, are also uniformly related to the invariants, and so suggest the introduction of a Lorentz group expansion. A triple Toller pole contribution can then be written down once the problem of incorporating the vertex covariance condition has been solved. (This covariance condition is really the origin of our problem since it is the covariance group, rather than the little group as in the $W=0$ problem, which changes its structure at the boundary we are considering). Such a pole will be a possible uniform asymptotic approximation to the amplitude, and corresponds to infinite sequences of triple Regge poles with the same trajectories in the expansions performed in the s-s region, the s-t region and on the boundary. So we conclude that a triple Toller pole provides a neat, but not essential solution to the analytic and asymptotic problems at this boundary. (The only essential requirement is that of analyticity, and a triple Regge pole can be made to satisfy this).

The basic problems in the introduction of analyticity into the group theoretic formalism have been studied in a sequence of papers [3-6] by Cosenza, Sciarrino and Toller, providing a powerful tool for the study of daughter problems and kinematic singularities and constraints. First they dealt with the problems for two-to-two amplitudes with various mass configurations. Subsequently, Toller [5] derived the fundamental results necessary for the introduction of a function of group theoretic variables which accurately reflects the analytic properties of a multiparticle amplitude. After stating some of the results obtained in ref. [1] in sect. 2, we briefly review the introduction of analyticity for the two-to-two amplitude, and its relation to the asymptotics at $W=0$, in sect. 3.

Sect. 4 contains the introduction of the analytic group variables for the three Reggeon vertex, which we still study in the context of the six particle spinless amplitude. In general, in order to introduce group variables for a particular amplitude it is necessary to define standard configurations [2, 7] for each of the vertices of the tree diagram considered. For convenience in the parametrization of the little groups these configurations are usually suitably aligned with the coordinate axes. At certain critical points (including $W=0$ and the boundary between the s-s and s-t regions) this alignment is not consistent with analyticity. At these points certain Gram determinants [8] formed from the external momenta vanish. Rather than these vectors spanning a lower dimensional space than usual, this corresponds, in general, to some vector becoming light-like or a plane becoming tangent to the light cone (that is, a plane becomes one spanned by a light-like and an orthogonal space-like vector). The introduction of analyticity can be pictured as the problem of rotating the corresponding standard vector or plane analytically in such a way that it becomes light-like or touches the light cone at the appropriate point. In ref. [1], we showed that it was convenient to take the standard triangle for the three-Reggeon vertex to be in the z, t plane in the s-t region and in the y, z plane in the s-s region. To make an analytic transition between the two regions it is necessary to rotate this triangle so that it touches the light cone at the boundary. Since, at this boundary the "mass" of one of the Reggeons is equal to the sum of the masses of the other two it is similar to a threshold or pseudo-threshold point. The problems are analogous, at least up to a complex transformation, to those considered by Cosenza, Sciarrino and Toller [4], at unequal mass threshold points for the two particle-Reggeon vertex. Unlike at these pseudo-threshold points, it is possible to keep the standard triangle (and the associated transformations) real in our problem. This makes it easier for us to adopt a geometric approach. Since the partial-wave expansion is performed on the real part of the little group and we discuss both the behaviour of this expansion and its asymptotic properties, it is important that this triangle can be kept real. The section concludes with a discussion of the uniform relation mentioned above, which exists between the analytic group variables and the invariants.

Sect. 5 consists of the mathematical preliminaries necessary for performing the Lorentz group expansion. This involves the decomposition of of the representation of $SL(2, \mathbb{C})^3$ in the coset space $SL(2, \mathbb{C})^3/SU(1,1)$. To do this we use a variation of the heuristic techniques introduced in ref. [1]. In sect. 6 we show how an expansion for the amplitude, in terms of representation functions of $SL(2, \mathbb{C})$, with the necessary covariance conditions, can be obtained. We arrange the standard triangle to move in such a way that a function on $SL(2, \mathbb{C})^3$ with a $SU(1,1)$ covariance condition, when restricted to the moving little groups, necessarily satisfies the vertex covariance condition for the amplitude. We can then use the expansion formula derived in sect. 5.

In sect. 7 we discuss the role of the three-Reggeon vertex in the group theoretic description of multiparticle amplitudes corresponding to an arbitrary tree diagram. We explain how such a diagram can always be extended to one containing only three-line vertices. Toller has used the results established in ref. [5] as the basis for a study of the analytic properties of the multi-Regge model for 2-to-n production processes. This model involves only "two-particle Reggeon" and "one particle two-Reggeon" vertices. When amplitudes involving six or more particles are considered diagrams containing three-Reggeon vertices [1, 9] become possible. The corresponding high energy limits are for processes having at least three particles in both the initial and final states. Although only of theoretical interest [10] such processes are important for example because of the multiparticle structure of the unitarity equations [8], and it is therefore desirable to understand the details of the Regge behaviour of such amplitudes in all possible limits. We describe the general process, for an arbitrary diagram, of introducing analytic group variables [5, 11], performing the $SU(1,1)$ expansion and finally an $SL(2, \mathbb{C})$ expansion.

2. THE THREE-REGGEON VERTEX

In ref. [1] we defined* the six particle amplitude as a function over the three little groups corresponding to fig. 1. We divided the momenta into three sets A, B and C and wrote

$$Q_A = \sum P_A , \quad Q_B = \sum P_B , \quad Q_C = \sum P_C , \tag{2.1}$$

and introduced standard configurations P_A^0, etc., in the (z, t) plane with sums Q_A^0, etc., along the z-axis. Then if

$$P_A = L(a_A)P_A^0 , \quad P_B = L(a_B)P_B^0 , \quad P_C = L(a_C)P_C^0 , \quad a_A, a_B, a_C \in \mathcal{L}_+ \tag{2.2}$$

* Unless otherwise stated the notation used is that of ref. [1].

and g_A, g_B, g_C are chosen so that

$$L(g_A)\,Q_A^o + L(g_B)\,Q_B^o + L(g_C)\,Q_C^o = 0\ , \tag{2.3}$$

we can define an element a so that

$$(Q_A,\ Q_B,\ Q_C) = L(a)\,(Q_A^{o'},\ Q_B^{o'},\ Q_C^{o'}) \tag{2.4}$$

where

$$Q_A^{o'} = L(g_A)\,Q_A^o \qquad \text{etc.}$$

Then

$$M(P_A,\ P_B,\ P_C)$$

$$= M(L(g_A h_A)\ P_A^o,\ L(g_B h_B)\,P_B^o,\ L(g_C h_C)\,P_C^o) \equiv f(h_A,\ h_B,\ h_C)\ , \tag{2.5}$$

where

$$h_A = g_A^{-1}\,a^{-1} a_A \in H_- \qquad \text{etc.}$$

Because the sets A, B, C contain just two particles we have the covariance conditions

$$f(h_A u_z(\nu_1),\ h_B u_z(\nu_2),\ h_C u_z(\nu_3)) = f(h_A,\ h_B,\ h_C)\ . \tag{2.6}$$

And also if $L(k)$ leaves the plane containing $Q_A^{o'}$, $Q_B^{o'}$, $Q_C^{o'}$ invariant

$$f(k\,h_A,\ k\,h_B,\ k\,h_C) = f(h_A,\ h_B,\ h_C)\ . \tag{2.7}$$

The group K of all such k is called the covariance group of the vertex.

It was necessary to distinguish two cases : (i) the s-s case in which the Q's lie in an entirely space-like plane; here g_A, etc., could be taken to act in the y, z plane and so K is isomorphic to S0(1,1). (ii) the s-t case in which the plane of the Q's contains time-like vectors, the g's can be chosen in the z, t plane and K is isomorphic to S0(2).

The partial-wave expansion took a different form in the two regions. In the s-t case

$$f(h_1,\ h_2,\ h_3)$$

$$= \int \sum_{n_1 n_2} \{F^{\Lambda_1\Lambda_2\Lambda_3}_{n_1,n_2}\ D^{\Lambda_1}_{n_1,0}(h_1)\ D^{\Lambda_2}_{n_2,0}(h_2)\ D^{\Lambda_3}_{-n_1-n_2,0}(h_3)\}\,\mathrm{d}\Lambda_1 \mathrm{d}\Lambda_2 \mathrm{d}\Lambda_3 \tag{2.8}$$

(where the n labels refer to the usual basis for the representation space in which rotations about the z-axis are diagonal) and in the s-s case it can be written in the form

$$f(h_1, h_2, h_3) = \int\int \mathrm{d}\kappa_1 \mathrm{d}\kappa_2 \sum_{\sigma_1\sigma_2\sigma_3} \{ F^{\Lambda_1\Lambda_2\Lambda_3}_{\sigma_1\sigma_2\sigma_3\kappa_1\kappa_2} D^{\Lambda_1}_{\sigma_1\kappa_1,0}(h_1)$$

$$\times D^{\Lambda_2}_{\sigma_2\kappa_2,0}(h_2) D^{\Lambda_3}_{\sigma_3(-\kappa_1-\kappa_2),0}(h_3)\} \mathrm{d}\Lambda_1 \mathrm{d}\Lambda_2 \mathrm{d}\Lambda_3 , \qquad (2.9)$$

where $\sigma = \pm 1$ and κ label a continuous SO(1,1) basis of the representation space [12].

A triple Regge pole contribution is then a triple pole in $F^{\Lambda_1\Lambda_2\Lambda_3}$ at $\Lambda_i = \alpha_i(Q_i^2)$. The asymptotic contribution of this pole to the amplitude is obtained by rewriting the expansions in terms of second-type representation functions and pulling back the contours in the Λ_i planes to pick up the pole contributions [7].

3. ANALYTICITY AND ASYMPTOTICS AT $W = 0$

Having defined the amplitude as a function of group theoretical variables, and discussed the corresponding hypothesis of Regge asymptotic behaviour, the next step is to consider how the analytic properties of the amplitude may be incorporated in this formalism. In this paper we will be concerned with the study of local analyticity properties, rather than the much more extensive problem of making sure that the full consequences of global properties, like cut-plane analyticity, are reflected in the coefficients of any partial-wave expansion. In particular, we want to examine whether a Regge-pole contribution has the right analyticity properties. If such a contribution is to be a uniform asymptotic approximation to the amplitude in some domain it should be free of singularities (apart from those required by unitarity) in the asymptotic region. The introduction of analyticity has been studied extensively and systematically by Cosenza, Sciarrino and Toller [3-6]. The conclusion is that it is necessary to modify the formalism at various critical points. For the unequal-mass two-to-two amplitude the most important point of this kind is where the square of the momentum transfer $W = Q^2 = 0$. Of course the problems associated with this point have been widely studied within various frameworks.

The singularities in a single-Regge-pole contribution at this point, which are cancelled by contributions from daughter poles in the usual approach, are intimately related to the bad behaviour of the asymptotic region in terms of the usual group variables. At the new type of critical point which occurs in the study of the three Reggeon vertex the behaviour of the asymptotic region is almost the reverse (and so has less significant implications). To illustrate this connection between the singularities and the asymptotic region, we briefly review the introduction of analyticity for the four-particle amplitude [3]. When analyticity is not being considered it is natural to take [1] the standard position of the momentum transfer, $Q^o = Q_W$, to be along the z-axis for each $W = Q^2 < 0$. Then $Q_W = (0, 0, 0, \sqrt{-W})$ which is singular at $W = 0$. This singularity has to be removed, for example by replacing Q_W by $\tilde{Q}_W = (\frac{1}{2}(1+W), 0, 0, \frac{1}{2}(1-W))$. We can then replace P_A^o and P_B^o by $\tilde{P}_A^o$ and $\tilde{P}_B^o$ respectively, still chosen in the z, t plane, with

$$\tilde{Q}_W = \sum \tilde{P}_A^o = \sum \tilde{P}_B^o \, , \tag{3.1}$$

and their members depending analytically on W at $W = 0$. The amplitude is then completely specified by a function

$$\tilde{f}(\tilde{h}) = M(\tilde{P}_A^o \, , \, L(\tilde{h})\tilde{P}_B^o) \, , \tag{3.2}$$

defined on $\tilde{H}_W^C$, the complexification of the little group of $\tilde{Q}_W$. If particles with spin are considered the amplitude may be reduced to a function $\tilde{f}_{m_1 m_2 m_3 m_4}(\tilde{h})$, bearing helicity labels corresponding to the external particles, defined over $\tilde{H}_W^C$ by a straightforward generalisation of this procedure, which we refer to in more detail in sect. 7. Then $\tilde{f}_{m_1 m_2 m_3 m_4}$ satisfies the covariance conditions

$$\tilde{f}_{m_1 m_2 m_3 m_4}(u_z(\mu)\,\tilde{h}u_z(\nu)) = e^{-i(m_2+m_4)\mu}\tilde{f}_{m_1 m_2 m_3 m_4}(\tilde{h})\, e^{-i(m_2+m_3)\nu} \tag{3.3}$$

It will have exactly the singularities that M has [5]. A Regge-pole contribution to $\tilde{f}$ will take the form of a second-type representation function of $\tilde{H}_W$ multiplied by a residue function. The various possibilities can be written down by choosing new axes so that $\tilde{H}_W$ becomes the usual SU(1,1) group, that is $H_W (= H_{Q^o})$. Such a change of coordinates will be specified by any Lorentz transformation c such that $L(c)Q_W = \tilde{Q}_W$; this gives a correspondence $\tilde{h} = c h c^{-1}$ between elements of $\tilde{H}_W$ and H_W and the usual parametrizations of H_W induce parametrizations of $\tilde{H}_W$. In order to conveniently incorporate the covariance condition (3.3), c should be chosen so that $L(c)$ maps the x, y plane onto the x, y plane. It is then uniquely specified up to a rotation in the x, y plane, which can only introduce phase factors in $\tilde{f}$. The relation between f and $\tilde{f}$ is

$$\tilde{f}(\tilde{h}) = f(c^{-1}\tilde{h}c) \, . \tag{3.4}$$

For definiteness we can take c to act only in the z, t plane and it then has the form

$$L(c) = \frac{1}{2\sqrt{-W}} \begin{pmatrix} 1-W & 1+W \\ 1+W & 1-W \end{pmatrix} . \quad (\text{columns } t, z) \tag{3.5}$$

The representation functions of $\tilde{H}_W$ will be those of H_W with argument $c^{-1}\tilde{h}c$ and so a Regge pole contribution to the amplitude will take the form

$$\beta^{A}_{m_1 m_3}(W)\,\beta^{B}_{m_2 m_4}(W)\,A^{-\alpha(W)-1}_{m_1+m_3,\,m_2+m_4}(c^{-1}\tilde{h}c)\ . \tag{3.6}$$

This is the same contribution as would be obtained from the non-analytic formulation. The only achievement of the analytic formulation in terms of examining the analyticity properties of a Regge-pole contribution has been to display the singularity at $W=0$ explicitly in c. In fact, if all the particles have zero spin, the simplest way to show that this is a singular contribution to the amplitude at this point, which cannot be made analytic by a suitable choice of the residue function, is to use invariants. But the SU(1.1) expansion only applies in $W<0$. The only finite singularities of the second-type function A occur at a set of points bounded in terms of its argument and the asymptotic region in which the contribution (3.6) can be expected (from the partial wave expansion) to dominate the amplitude is that for which the argument of the A- function is large. Because for $W<0$, c is analytic, the contribution (3.6) will be analytic apart from the singularities at a bounded set of points due to those of the A-function and so may provide a uniform asymptotic approximation to the amplitude in any compact set in $W<0$. However, H^{C}_{0} is not isomorphic to H^{C}_{W} [3] and so c must diverge as $W\to 0$. (The fact that these groups are not isomorphic may be viewed as the fundamental problem). Therefore (3.6) cannot provide an asymptotic approximation (in the limit $L(\tilde{h})\to\infty$) uniform for W in any set including $W=0$. If we had this sort of uniform behaviour in such a set it would also show up at $W=0$. In fact the size of $\tilde{h}$ is uniformly related to the invariants.

The relation between $z=\cosh\zeta$, where ζ is the usual boost angle corresponding to h, and the invariant $s=(P_1+P_2)^2$ is given by

$$s=\phi_1(M_i)+\phi_2(M_i)z-\frac{z-1}{2W}(M_3^2-M_1^2)(M_4^2-M_2^2)+\mathrm{O}(W)\ , \tag{3.7}$$

where ϕ_1, ϕ_2 are functions of the masses only. Fixed values of z correspond to infinitely increasing values of s as $W\to 0$. Thus in this limit the asymptotic region where the pole contribution can be expected to dominate recedes to infinity. The size of $\tilde{h}$ can be described by $\tilde{z}=\cosh\tilde{\zeta}$ where $\tilde{\zeta}$ is the boost angle in SL(2, $\mathbb{C}$) corresponding to $\tilde{h}$(i.e. $\tilde{h}=u_1 a_z(\tilde{\zeta})u_2$ for some u_1, $u_2\in$ SU(2)) z and $\tilde{z}$ are related by

$$(z-1)=\frac{-4(\tilde{z}-1)W}{(1-W)^2}\ , \tag{3.8}$$

and so if we substitute this into (3.7) we see that s depends uniformly on $\tilde{z}$ in the limit $W\to 0$. [Since, at fixed W, $\tilde{f}$ is defined over $\tilde{H}^{C}_{W}$, in general it is defined over the set $\{(W,\tilde{h}):\ \tilde{h}\in\tilde{H}^{C}_{W}\}$. As, $H^{C}_{W}\subseteq \mathrm{SL}(2,\mathbb{C})^{C}$ we may try to induce a complex analytic manifold structure on the set from that of SL(2, $\mathbb{C}$)^{C}. This can obviously be done in a neighbourhood of any $W\neq 0$. The parametrization of μ, $\tilde{\zeta}$, ν of $\tilde{H}_W$, $W\neq 0$

$$\begin{pmatrix} e^{-\frac{1}{2}i\mu} & 0 \\ 0 & e^{\frac{1}{2}i\mu} \end{pmatrix} \frac{1}{1-W}$$

$$\times \begin{pmatrix} [(1-W)^2 - 4W\sinh^2\frac{1}{2}\tilde{\zeta}]^{\frac{1}{2}} & 2\sinh\frac{1}{2}\tilde{\zeta} \\ -2W\sinh\frac{1}{2}\tilde{\zeta} & [(1-W)^2 - 4W\sinh^2\frac{1}{2}\tilde{\zeta}]^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} e^{-\frac{1}{2}i\nu} & 0 \\ 0 & e^{\frac{1}{2}i\nu} \end{pmatrix}, \tag{3.9}$$

makes it possible to see that this can also be done about $W=0$. The little group $\tilde{H}_0 \cong E(2)$ is obtained by putting $W=0$ in (3.9). The contribution of (3.6) contains the second type function [2] $a^{-\alpha(W)-1}_{m_1+m_3,\, m_2+m_4}(\zeta)$ where ζ is related to the analytic coordinates W, $\tilde{\zeta}$ by (3.8). This function then has an essential singularity at $W=0$ (for each $\tilde{\zeta}$).] The solution of Cosenza, Sciarrino and Toller [3] for the problems at $W=0$ remedies the difficulty due to the receding asymptotic region for Regge pole contributions. They show that if it is assumed that the function $\tilde{f}(\tilde{h})$ can be extended to a function over the whole of SL(2, $\mathbb{C}$) with (3.3) still holding throughout the group; then the partial wave expansion can be performed in terms of representation functions of SL(2, $\mathbb{C}$). The contribution of a Lorentz or Toller pole will then be expected to dominate the amplitude at large value of $\tilde{h}$ instead of large values of h. A Toller pole can be decomposed into an infinite set of Regge poles with parallel trajectories [14]. Consequently such a contribution shows Regge behaviour at $W=0$ and can give a uniform asymptotic approximation to the amplitude in a neighbourhood of this point.

Of course a Toller pole is not the only solution to the difficulties at $W=0$. but it does exploit the introduction of analyticity into the group theoretical approach to complex angular momentum.

4. ANALYTICITY AND ASYMPTOTICS AT THE BOUNDARY OF THE s-t AND s-s REGIONS ($\lambda_{abc} = 0$)

A triple Regge pole contribution to the six-particle spinless amplitude in an s-t region takes the form

$$\beta_{n_1, n_2, -n_1-n_2}(Q_A^2, Q_B^2, Q_C^2)\, e^{-i(n_1[\mu_A - \mu_C] + n_2[\mu_B - \mu_C])}$$

$$a^{-\alpha_A(Q_A^2)-1}_{n_1, o}(\zeta_A)\, a^{-\alpha_B(Q_B^2)-1}_{n_2, o}(\zeta_B)\, a^{-\alpha_C(Q_C^2)-1}_{-n_1-n_2, o}(\zeta_C)\,. \tag{4.1}$$

We consider the analyticity of this expression in the neighbourhood of the boundary between the s-t and s-s regions, say $\sqrt{-Q_A^2} = \sqrt{-Q_B^2} + \sqrt{-Q_C^2}$. A simple method of doing this is to use invariants. For example ζ_A is related to $u_A = (P_2 + Q_B)^2$ by the equation

$$u_A = Q_B^2 + M_2^2 - \frac{1}{4Q_A^2}\{[\lambda(Q_A^2, Q_B^2, Q_C^2)\lambda(M_1^2, M_2^2, Q_A^2)]^{\frac{1}{2}} \cosh \zeta_A$$

$$- (M_1^2 - M_2^2 + Q_A^2)(Q_B^2 - Q_C^2 + Q_A^2)\}, \qquad (4.2)$$

where

$$\lambda(a^2, b^2, c^2) = (a+b+c)(c-a-b)(b-c-a)(a-b-c) \quad . \qquad (4.3)$$

Thus the relation between u_A and $\cosh \zeta_A$ is singular on the boundary, where $\lambda_{abc} \equiv \lambda(Q_A^2, Q_B^2, Q_C^2) = 0$, as would be expected. A similar but more complicated expression exists for μ_A, but it is not singular at $\lambda = 0$. If we continue (4.1) around the boundary (at fixed u_A, etc) in a suitable direction $\cosh \zeta_A$ goes to $-\cosh \zeta_A$ via the lower half complex plane. This only has the effect of introducing a phase factor $\exp[-\pi(\alpha_A(Q_A^2) + \alpha_B(Q_B^2) + \alpha_C(Q_C^2))]$ because of a symmetry property of the a^j_{mn} functions.*

This can be removed by a branch point in β. This is not enough to guarantee that (4.1) does not have a pole or essential singularity on the boundary. To do this we use an asymptotic formula for a^j_{mn}

$$a^j_{m,n}(\zeta) = \frac{(-1)^{m-n}\Gamma(-2j-1)}{\Gamma(-j-m)\Gamma(-j+m)} z^{-j-1}(1 + O(|z|^{-2})) \; . \qquad (4.4)$$

If

$$\beta = \beta' \lambda_{abc}^{\frac{1}{2}(\alpha_A + \alpha_B + \alpha_C) + N}(1 + O(|\lambda|)) \; . \qquad (4.5)$$

where N is an integer and $\beta' = \beta'(Q_A^2, Q_B^2, Q_C^2)$ is regular on $\lambda_{abc} = 0$, (4.1) will have a pole of order $(-N)$ on $\lambda_{abc} = 0$ for N negative, and will be zero if N is positive. If N is zero the leading term in the asymptotic expansion will still have the Regge form in terms of invariants, that is it will be a product of factors of the form $((u_A - f_{1A})/f_{2A})^{\alpha_A}$ where f_{iA} is a function of the masses and the variables Q^2. But at $\lambda_{abc} = 0$ the other non-leading terms will vanish and so the leading term will represent the entire contribution of a triple Regge pole at this point. This contribution has been obtained from the partial wave expansion in the s-t region, where $\lambda_{abc} > 0$, and continued to $\lambda_{abc} = 0$. If we were to perform the expansion at $\lambda_{abc} = 0$, with the appropriate covariance condition, then a single triple Regge pole

*The symmetry property of the a^j_{mn} function required is that it acquires a phase factor $e^{i\pi j}$ when continued from $z > 1$ in the lower half complex z plane to $-z$: $a^j_{mn}(-z) = e^{i\pi j} a^j_{mn}(z)$ (See, for example, M. Andrews and J. Gunson: Journ. Math. Phys., 5 1391, (1965)). We are using the form for a^j_{mn} given in ref. 14, the asymptotic formula (4.4) follows from properties of the hypergeometric function.

in this expansion, would contribute the appropriate second-type representation function. This analysis could be performed by choosing the standard triangle formed by the Q's to lie in a plane containing the z-axis and a light-like vector in the (y, t) plane. This would leave the little groups unchanged, although it would of course give a new covariance group. With the triangle suitably chosen the relation between u_A and the boost angle $\tilde{\zeta}_A$ would be given by (4.2) with λ_{abc} replaced by one. So the dependence on the boost angles $\tilde{\zeta}_A$ etc., of a non-singular, non-zero, Regge-pole contribution continued from the s-t region, will be a product of factors of the form $(\cosh \tilde{\zeta}_A)^{\alpha}$. It can be shown that this is of the form of a second-type representation function.

Thus we have shown that a single triple-Regge-pole contribution from the s-t region can be made analytic at $\lambda_{abc} = 0$. This depends on the continuation along a path encircling $\lambda_{abc} = 0$; another problem, which we will discuss below, is to continue half way round this point and relate Regge pole contributions in the s-s and s-t regions. However the analyticity of s-t region contributions is the most important consideration because these regions adjoin the physical regions where Regge-pole trajectories will produce physical particles.

In sect. 3, we discussed the relation between the singularity of a Regge pole contribution to the four particle amplitude at $W=0$ and the unsatisfactory behaviour of the asymptotic region. Similarly the bad behaviour of the triple Regge pole contribution (4.1) at $\lambda_{abc} = 0$ is reflected in the behaviour of the asymptotic region. The region in which, from the partial wave expansion, the contribution (4.1) can be expected to dominate the amplitude, is that in which ζ_A, ζ_B, ζ_C are large. Whereas equation (4.2) shows that as $\lambda_{abc} \to 0$ with ζ_A kept fixed, u_A will tend to the same fixed value independent of ζ_A. In consequence, the asymptotic region 'comes in' to finite values of the invariants. On the basis of this we would expect a contribution containing factors of the form $((u_A - f_{1A})/f_{2A})^{\alpha_A}$ to dominate at $u_A = f_{1A}$ (we can let $\lambda_{abc} \to 0$, $u_A \to f_{1A}$ and keep ζ_A large) where it may either have a pole or be zero. Of course, what can be expected to happen is that the background grows in importance. So the partial-wave expansion based on the present group variables is not a satisfactory basis for discussing asymptotic behaviour near this point.

In order to make a smooth transition from the s-t to the s-s region it is necessary to continuously rotate the standard plane of the Q's. So, instead of the standard triangle $(Q_A^{o\prime}, Q_B^{o\prime}, Q_C^{o\prime})$ in the (y, z) or (z, t) plane, that we used before, we take a standard triangle $(\tilde{Q}_A^o, \tilde{Q}_B^o, \tilde{Q}_C^o)$ in the plane containing the z-axis and the vector $(1 + \lambda_{abc}, 0, 1 - \lambda_{abc}, 0)$ with sides of the appropriate lengths. Because $\lambda_{abc} \gtrless 0$ corresponds to the s-t and s-s cases respectively, we can choose the Q's to be real analytic functions of the Q_A^2, Q_B^2, Q_C^2 (for negative Q_A^2 etc., in a neighbourhood of any point on $\lambda_{abc} = 0$). Further it is possible to choose $\tilde{P}_A^o$, $\tilde{P}_B^o$, $\tilde{P}_C^o$, standard configurations for the sets of external particles, to depend analytically on Q_A^2, Q_B^2, Q_C^2, and such that $\sum \tilde{P}_A^o = \tilde{Q}_A^o$, etc. If $a, a_A, a_B, a_C \in SL(2, C)$

are chosen so that

$$(Q_A,\ Q_B,\ Q_C) = L(a)(\tilde{Q}_A^o,\ \tilde{Q}_B^o,\ \tilde{Q}_C^o)\ , \tag{4.6}$$

$$P_A = L(a_A)\,\tilde{P}_A^o\ , \qquad P_B = L(a_B)\,\tilde{P}_B^o\ , \qquad P_C = L(a_C)\,\tilde{P}_C^o\ , \tag{4.7}$$

we can replace $M(P_A, P_B, P_C)$ by a function of Q_A^2, Q_B^2, Q_C^2 and the elements of the little groups $\tilde{H}_A, \tilde{H}_B, \tilde{H}_C$ of $\tilde{Q}_A^o$, $\tilde{Q}_B^o$, $\tilde{Q}_C^o$ respectively, defined as follows:

$$\begin{aligned} M(P_A,\ P_B,\ P_C) &= M(L(a_A)\,\tilde{P}_A^o\ ,\ L(a_B)\tilde{P}_B^o\ ,\ L(a_C)\,\tilde{P}_C^o) \\ &= M(L(a^{-1}a_A)\,\tilde{P}_A^o,\ L(a^{-1}a_B)\,\tilde{P}_B^o,\ L(a^{-1}a_C)\,\tilde{P}_C^o) \\ &\equiv f(\tilde{h}_A,\ \tilde{h}_B,\ \tilde{h}_C)\ , \end{aligned} \tag{4.8}$$

where

$$\tilde{h}_A = a^{-1}a_A \in \tilde{H}_A\ , \qquad \text{etc.}\ldots \tag{4.9}$$

(Again we have spinless particles for simplicity; for particles with spin $\tilde{f}$ bears helicity labels corresponding to the external particles.) The function $\tilde{f}$ is thus defined over the analytic manifold $\{(Q_A^2,\ Q_B^2,\ Q_C^2,\ \tilde{h}_A,\ \tilde{h}_B,\ \tilde{h}_C)$: $\tilde{h}_A \in \tilde{H}_A,\ \tilde{h}_B \in \tilde{H}_B,\ \tilde{h}_C \in \tilde{H}_C\}$ and by theorems of Toller [4] will have exactly the same singularities as M.

In order to perform a partial-wave expansion of $\tilde{f}$ we have to parametrize the moving groups $\tilde{H}_A$, $\tilde{H}_B$, $\tilde{H}_C$, effectively by mapping them on to the standard SU(1,1) group. One way of doing this would be to make this analytic formulation more like our non-analytic one by introducing elements $\tilde{g}_A$, $\tilde{g}_B$, $\tilde{g}_C$ which act in the plane of the $\tilde{Q}$'s, depend analytically on the Q^2's and have the properties that $L(\tilde{g}_A)\,\tilde{Q}_A^o$, $L(\tilde{g}_B)\tilde{Q}_B^o$ and $L(\tilde{g}_C)\,\tilde{Q}_C^o$ are along the z-axis. Then $\tilde{h}_A \to \tilde{g}_A\tilde{h}_A\tilde{g}_A^{-1}$ maps $\tilde{H}_A$ onto SU(1,1) (the little group of the z-axis). However $\tilde{f}$ has to satisfy the covariance conditions

$$\tilde{f}(\hat{k}\tilde{h}_A,\ \hat{k}\tilde{h}_B,\ \hat{k}\tilde{h}_C) = \tilde{f}(\tilde{h}_A,\ \tilde{h}_B,\ \tilde{h}_C)\ , \quad \hat{k} \in K_{\lambda abc}\ , \tag{4.10}$$

where $K_{\lambda abc}$ is the subgroup of SL(2, $\mathbb{C}$) which leaves the plane of the $\tilde{Q}^o$'s unchanged, and

$$\tilde{f}(\tilde{h}_A\hat{u}_A,\ \tilde{h}_B\hat{u}_B,\ \tilde{h}_C\hat{u}_C) = \tilde{f}(\tilde{h}_A,\ \tilde{h}_B,\ \tilde{h}_C)\ ,\quad \hat{u}_A \in \hat{U}_A,\ \hat{u}_B \in \hat{U}_B,\ \hat{u}_C \in \hat{U}_C\ , \tag{4.11}$$

where $\hat{U}_A$ is the subgroup of SL(2, $\mathbb{C}$) which leaves the plane of $\tilde{P}_A^o$ unchanged, and similarly for B and C. Thus even if we introduce the $\tilde{g}$'s to obtain fixed little groups we will have varying covariance conditions. Varying covariance conditions present no problem whilst we can find an element of the little group, varying analytically, which maps the covariance plane onto a convenient fixed plane (i.e. one determined by two of the axes of the group). This is clearly possible for the conditions of (4.11); we can find such an element $\tilde{d}_A \in SU(1.1)$ which maps $L(\tilde{g}_A)\tilde{P}_A^0$ into P_A^0. For $\lambda_{abc} > 0$ we can also find $r \in SU(1,1)$ which maps the plane of the $\tilde{Q}^0$'s back on the (z,t) plane. We can choose $L(r)$ to act in the (y,t) plane only and then:

$$L(r) = \frac{1}{2\sqrt{\lambda_{abc}}}\begin{pmatrix} \lambda_{abc}+1 & \lambda_{abc}-1 \\ \lambda_{abc}-1 & \lambda_{abc}+1 \end{pmatrix} . \tag{4.12}$$

The partial wave expansion will now take the form

$$f(\tilde{h}_A, \tilde{h}_B, \tilde{h}_C) = \int d\Lambda \sum_{n_A+n_B+n_C=0} F^{\Lambda_A\Lambda_B\Lambda_C}_{n_A n_B n_C} \prod_{X=A,B,C} D^{\Lambda_X}_{n_X,0}(r\tilde{g}_X\tilde{h}_X\tilde{g}_X^{-1}\tilde{d}_X^{-1}) . \tag{4.13}$$

Thus in the s-t region we have effectively regained the 'non-analytic' analysis much as we did in sect. 3. $L(r)$ is singular at $\lambda_{abc} = 0$ and complex in the s-s region where $\lambda_{abc} < 0$. Thus all we have achieved in (4.13) is to make explicit the singularity of the expansion continued from the s-t region at $\lambda_{abc} = 0$. We have shown that the corresponding singularity in a single triple-Regge-pole contribution can be removed by appropriate behaviour of the residue function.

However, the covariance group $\hat{K}_{\lambda_{abc}}$ as a subgroup of the complexification of $\tilde{H}_A \times \tilde{H}_B \times \tilde{H}_C$ changes its structure at $\lambda_{abc} = 0$, and so we cannot expect to continue the partial-wave expansion to this point. This change of structure has been discussed explicitly by Cosenza, Sciarrino and Toller in appendix D of ref. [3]. They considered a pseudo-threshold for the four particle amplitude, but as we remarked in the introduction, this is exactly analogous to our problem.

When we attempt to continue (4.13) into $\lambda_{abc} < 0$ the arguments of the representation functions become complex, covergence is no longer obvious, and its relation to the partial wave expansion in the s-s region is not immediate. By writing the representation functions in (4.13) as matrix products the part which depends on r and continues into a function of pure imaginary argument may be separated out. It is clear that some restrictions on the partial-wave coefficients F are necessary to perform this continuation and so it may only be performed for a certain class of analytic square integrable functions in the s-t region. In particular it can be easily effected for some Regge pole contributions, that is those whose residues are non-zero for only finitely many values of the helicity labels. Such contributions will, however, necessarily fail to be square integrable over the coset space $SU(1,1)^3/K$ but this is not obviously unreasonable physically.

We can now show that the asymptotic region in terms of the analytic group variables, $\tilde{h}_A$, etc., is uniformly related to the invariants near $\lambda_{abc} = 0$. Since we chose the $\tilde{g}_A$ analytically a convenient analytic choice of the $\tilde{P}^0_A$ is $L(\tilde{g}_A^{-1}) P^0_A$, so that $\tilde{d}_A$ can be taken to be the identity. Then $\tilde{g}_A \tilde{h}_A \tilde{g}_A^{-1} \in SU(1,1)$. One way of choosing the standard moving triangle of the $\tilde{Q}$'s is to align $\tilde{Q}^0_A$ along the z-axis, all of the time. This means that $\tilde{g}_A$ will also be the identity and so $\tilde{P}^0_A = P^0_A$. The invariant u_A is then related to $\tilde{h}_A$ by

$$u_A = (\tilde{Q}^0_B + L(\tilde{h}_A) P^0_{A1})^2 \tag{4.14}$$

If we parametrize $\tilde{H}_A$ by

$$\tilde{h}_A = \hat{k}_A a_x(\tilde{\zeta}) \hat{u}_A , \tag{4.15}$$

where $\hat{u}_A \in \hat{U}_A$, $\hat{k}_A \in \hat{K}$ and, as usual, $a_x(\tilde{\zeta})$ is a boost along the x-axis then

$$u_A = (\tilde{Q}^0_B + L(a_x(\tilde{\zeta})) P^0_{A1})^2 \tag{4.16}$$

since

$$L(\hat{k}_A^{-1}) \tilde{Q}^0_B = \tilde{Q}^0_B , \quad \text{and } L(u_A) P^0_{A1} = P^0_{A1} . \tag{4.17}$$

Also since $\tilde{Q}^0_B = L(r^{-1}) Q^0_B$ we have that

$$u_A = (Q^0_B + L(r a_x(\tilde{\zeta})) P^0_{A1})^2 . \tag{4.18}$$

So u_A is given by (4.2) where ζ_A is the boost angle of $r a_x(\tilde{\zeta})$ when expressed in terms of the usual parametrization of SU(1,1). The relation between $\cosh \zeta_A$ and $\cosh \tilde{\zeta}_A$ is given by

$$(\lambda_{abc}^{\frac{1}{2}} + \lambda_{abc}^{-\frac{1}{2}}) \cosh \tilde{\zeta}_A = 2 \cosh \zeta_A \tag{4.19}$$

and so the relation between u_A and $\cosh \tilde{\zeta}_A$ is uniform. The parametrization we have used of $\tilde{H}_A$ has been so arranged, to factor out the dependence on the covariance groups on either side, to obtain a correctly adjusted measurement of the size of $\tilde{h}_A$. This adjustment is obviously necessary because of the irrelevance of the size of the boost in the covariance group in the s-s region.

For this calculation we have used $\tilde{g}_A = e$, but in cases where the corresponding $\tilde{g}$ is not the identity it will still be analytic and so we can set up coordinates in SU(1.1), of the form used above, for $\tilde{g}_A \tilde{H}_A \tilde{g}_A^{-1}$ and these will induce analytic co-ordinates of $\tilde{H}_A$ which will have suitable asymptotics.

So to obtain a uniform description of the asymptotic region in the neighbourhood of $\lambda_{abc} = 0$, it is again (as at $W = 0$ in sect. 3) necessary to use little group variables moving analytically within $SL(2, \mathbb{C})^3$. Therefore, to obtain an asymptotic expansion of the amplitude which is well-behaved in this neighbourhood, we should try to extend $\tilde{f}(\tilde{h}_A, \tilde{h}_B, \tilde{h}_C)$ to a function over $SL(2, \mathbb{C})^3$ in a suitable manner. Before we can do this it is necessary to consider the technical details of the expansion of such a function, which we do in the next section.

5. HARMONIC ANALYSIS IN THE COSET SPACE $SL(2, \mathbb{C})^3/SU(1,1)$

In the last section we gave reasons for considering functions on $SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ and in the next section we shall show that the appropriate covariance condition to impose on such a function is

$$f(va_1, va_2, va_3) = f(a_1, a_2, a_3) \;, \qquad v \in SU(1,1) \;, \qquad a_i \in SL(2, \mathbb{C}) \;. \tag{5.1}$$

Of course, there will also be compact covariance conditions on the right but these will cause no essential difficulty. To obtain the expansion of such a function in terms of representation functions of $SL(2, \mathbb{C})$, it is necessary to decompose the continuous unitary representation of $SL(2, \mathbb{C})^3$ which acts in the space of functions on the coset (or homogeneous) space $SL(2, \mathbb{C})^3/SU(1,1)$, square integrable with respect to the corresponding invariant measure. The existence of this invariant measure is immediate since we are considering the quotient space of two unimodular groups *. From a well known theorem [15, 16] this representation can be decomposed into a direct integral of irreducible unitary representations. Since $SU(1,1)$ is non-compact the decomposition of this representation is not easily deducible from the decomposition of the regular representation of $SL(2, \mathbb{C})^3$. But we can attempt to incorporate the covariance conditions in the partial-wave coefficients of the regular representation in terms of their being parallel to certain non-normalisable vectors in each representation space. We can then make a heuristic deduction of the required decomposition formula from that of the regular representation. Such techniques for similar problems have been given in ref. [1] and the resulting expansion formulae proved directly for the cases to which they were applied; that is $SU(1,1)/SO(1,1)$ and $SU(1,1)^3/SO(1,1)$. We now give a similar method for this case.

We saw in ref. [1] that it is convenient to choose a basis for the representation in which the appropriate subgroup is represented by diagonal matrices. The usual form in which the representation matrices of $SL(2, \mathbb{C})$, corresponding to the representation (λ, M) are given, is $\mathcal{D}^{\lambda M}_{jmj'm'}(a)$ [14],

*See appendix B of ref. [1].

where λ is pure imaginary, M is a non negative integer or half integer, and j and m are discrete labels, labelling an 'SU(2) basis' ($j \geqslant M$, $j \geqslant |m|$ and $j-M$, $j-m$ are integers). Alternatively, it is possible to use an SU(1,1) (pseudo) basis [16, 17]. This basis would be labelled by (τ, l, m) where $\tau = \pm 1$, l runs over the irreducible unitary representations of SU(1,1) and m runs over the appropriate values for that representation (Here, of course, l is used as an abbreviation for (ϵ, l), $\epsilon = 0, \frac{1}{2}$, $\mathrm{Re}\, l = -\frac{1}{2}$, $\mathrm{Im}\, l > 0$ or $(k, \pm)$, $k = 0, \frac{1}{2}, 1, \ldots.$ and only (ϵ, l) where $M-\epsilon$ is an integer, and $(k, \pm)$, where $k \leqslant M$ and $k-M$ is an integer, are required for the representation (λ, M)). Because we again have covariance conditions of different characters for the two sides, it is most convenient to use an SU(2) basis on the right hand side of the representation function and an SU(1,1) basis on the left; that is we are using two different bases in the representation space. Instead of realising the representation spaces by the labelling of bases, we can take them to be the space of square integrable functions $f(v, \tau), v \in \mathrm{SU}(1,1)$, $\tau = \pm 1$ satisfying the covariance condition $f(u_z(\mu)\, v, \tau) = \mathrm{e}^{-iM\mu\tau} f(v, \tau)$. This realisation can be used instead of the SU(1,1) basis on the left, and the representation (λ, M) is then determined by functions $\mathcal{D}^{\lambda M}_{v\tau jm}(a)$, and its action is to send the vector, with coordinates ϕ_{jm} in the SU(2) basis, into the function $f(v, \tau)$ in the above space of square integrable functions where

$$f(v,\tau) = \sum_{jm} \mathcal{D}^{\lambda M}_{v\tau jm}(a)\, \phi_{jm} \,. \tag{5.2}$$

In order to understand the sort of problems that arise, it is perhaps preferable to consider first the problem of $\mathrm{SL}(2, \mathbb{C})^2/\mathrm{SU}(1,1)$. For a function f defined on $\mathrm{SL}(2, \mathbb{C})^2$ we can hope to define partial wave coefficients

$$F^{\lambda_i M_i}_{v_i \tau_i j_i m_i} \equiv F^{\lambda_1 M_1\, \lambda_2 M_2}_{v_1 \tau_1 j_1 m_1\, v_2 \tau_2 j_2 m_2}\,,$$

by

$$F^{\lambda_i M_i}_{v_i \tau_i j_i m_i} = \int f(a_1, a_2) \prod_{i=1,2} \mathcal{D}^{\lambda_i M_i}_{v_i \tau_i j_i m_i}(a_i)\, \mathrm{d}a_i \,. \tag{5.3}$$

Since $\mathcal{D}^{\lambda M}_{v\tau v'\tau'}(v_0) = \delta_{\tau\tau'}\delta(v v_0 v'^{-1})$ the covariance condition $f(va_1, va_2) = f(a_1, a_2)$, $v \in \mathrm{SU}(1,1)$ is exactly equivalent to the condition $F^{\lambda_i M_i}_{\tau_i v, \tau_i, j_i m_i} = F^{\lambda_i M_i}_{v_i, \tau_i, j_i m_i}$, $v \in \mathrm{SU}(1,1)$ on f's partial wave coefficients. If we abbreviate $F^{\lambda_i M_i}_{v_i \tau_i j_i m_i}$ to $\psi(v_1, v_2)$ the property $\psi(v_1 v, v_2 v) = \psi(v_1, v_2)$ means that ψ is determined by $\psi(e, v)$ and squared integrability of f over the coset space $\mathrm{SL}(2, \mathbb{C})^2/\mathrm{SU}(1,1)$ will correspond to square integrability of $\psi(e, v)$ over SU(1,1) rather than ψ over $\mathrm{SU}(1,1)^2$. In this case we can expand ψ in terms of representation functions of SU(1,1)

$$\psi(e,v) = \int \sum_{mm'} \overline{\Psi^{l}_{mm'}}\, D^{l}_{mm'}(v)\, \mathrm{d}l \,. \tag{5.4}$$

Because of the covariance property satisfied by the functions forming the representation spaces $\psi(u_z(\mu_1)v_1,\ u_z(\mu_2)v_2) = e^{-i(M_1\tau_1\mu_1+M_2\tau_2\mu_2)}$ and so $\Psi_{mm'} = 0$ unless $m = +M_2\tau_2$ and $m' = -M_1\tau_1$. The coefficients of f, using the SU(1,1) basis on the left, $F^{\lambda_i M_i}_{l_i m_i', j_i m_i}$ are related to $F^{\lambda_i M_i}_{v_i\tau_i, j_i m_i}$ by

$$F^{\lambda_i M_i}_{v_i\tau_i,\, j_i m_i} = \sum_{m_i'} \int \mathrm{d}l_i\ F^{\lambda_i M_i}_{l_i m_i',\, j_i m_i} \prod_{i=1,2} D^{l_i}_{\tau_i M_i,\ m_i'}(v_i) \quad . \tag{5.5}$$

Since $\overline{D^l_{m,m'}(v)} = D^l_{-m,-m'}(v)$ we have

$$D^l_{\tau_2 M_2,\, -M_1\tau_1}(v_2 v_1^{-1}) = \sum_m D^l_{\tau_1 M_1, m}(v_1)\, D^l_{\tau_2 M_2,\, -m}(v_2)\ , \tag{5.4}$$

and so from (5.4) we can write

$$F^{\lambda_i M_i}_{v_i\tau_i,\, j_i m_i} = \sum_m \int \mathrm{d}l\ \mathcal{F}^{\lambda_i M_i}_{l,\tau_i,\, j_i m_i}\, D^l_{\tau_1 M_1, m}(v_1)\, D^l_{\tau_2 M_2,\, m}(v_2)\ , \tag{5.6}$$

and so

$$F^{\lambda_i M_i}_{\tau_i l_i m_i' j_i m_i'} = \mathcal{F}^{\lambda_i M_i}_{l,\tau_i,\, j_i m_i}\, \delta(l_1 - l_2)\, \delta_{m_1',\, -m_2'}\ . \tag{5.7}$$

An alternative way of seeing that this is the result to expect, is as follows:

$$F^{\lambda_i M_i}_{l_i\tau_i m_i' j_i m_i} = \int f(a_1, a_2) \prod_{i=1,2} \overline{\mathcal{D}^{\lambda_i M_i}_{\tau_i l_i m_i' j_i m_i}(a_i)}\, \mathrm{d}a_i\ . \tag{5.8}$$

If $[a_1, a_2]$ denotes the coset of (a_1, a_2) in SL(2, ℂ)^2/SU(1,1) we can factor the invariant measures $\mathrm{d}a_1\mathrm{d}a_2 = \mathrm{d}[a_1, a_2]\,\mathrm{d}v$ and

$$F^{\lambda_i M_i}_{\tau_i l_i m_i' j_i m_i} = \int f([a_1, a_2]) \sum_{m_i''} \prod_{i=1,2} \mathcal{D}^{\lambda_i M_i}_{\tau_i l_i m_i'' j_i m_i}(a_i)$$

$$\times \int \prod_{i=1,2} D^{l_i}_{m_i'' m_i'}(v)\ \mathrm{d}v\ \mathrm{d}[a_1, a_2]\ . \tag{5.9}$$

So using

$$\int \prod_{i=1,2} D^{l_i}_{m_i'' m_i}(v)\ \mathrm{d}v = \delta(l_1 - l_2)\, \delta_{m_1'',\, -m_2''}\, \delta_{m_1,\, -m_2}\quad , \tag{5.10}$$

we obtain (5.7) with

$$\mathcal{F}^{\lambda_i M_i}_{l,\tau_i,j_i m_i}$$

$$= \int f([a_1, a_2]) \sum_{m'} \mathcal{D}^{\lambda_i M_i}_{\tau_1 l, m', j_1' m_1'}(a_1)\, \mathcal{D}^{\lambda_i M_i}_{\tau_2 l, -m', j_2' m_2'}(a_2)\, \mathrm{d}[a_1, a_2] \quad (5.11)$$

In the problem that really concerns us, of $SL(2, \mathbb{C})^3/SU(1,1)$ we can proceed similarly and define coefficients $F^{\lambda_i M_i}_{\tau_i v_i j_1' m_1'}$ abbreviated to $\psi(v_1, v_2, v_3)$ satisfying $\psi(v_1 v, v_2 v, v_3 v) = \psi(v_1, v_2, v_3)$ if f satisfies (5.1). Hence we can expand $\psi(e, v_2', v_3')$ in terms of functions $D^{l_2}_{M_2 \tau_2, \mu}(v_2') D^{l_3}_{M_3 \tau_3, -M_1 \tau_1 - \mu}(v_3')$. Consequently $\psi(v_1, v_2, v_3)$ can be expanded in terms of

$$\sum_{m,m'} D^{l_2}_{-\mu,-m}(v_1)\, D^{l_3}_{\tau_1 M_1 + \mu, -m'}(v_1)\, D^{l_2}_{\tau_2 M_2, m}(v_2)\, D^{l_3}_{\tau_3 M_3, m'}(v_3) \quad (5.12)$$

Now we can use the SU(1,1) Clebsch-Gordan coefficients [18, 19] to write

$$D^{l_2}_{-\mu,-m}(v_1)\, D^{l_3}_{\tau_1 M_1+\mu, -m'}(v_1)$$

$$= \int \mathrm{d}l_1\, \overline{C^{l_2 l_3 l_1}_{-\mu, \tau_1 M_1+\mu, \tau_1 M_1}}\, C^{l_2 l_3 l_1}_{-m, -m', m+m'}\, D^{l_1}_{\tau_1 M_1, m+m'}(v_1)\ , \quad (5.13)$$

and so

$$F^{\lambda_i M_i}_{\tau_i l_i m_i', j_i m_i} = \sum \mathcal{F}^{\lambda_i M_i}_{l_1 l_2 l_3 \tau_i j_i m_i}\, C^{l_2 l_3 l_1}_{-m_2', -m_3', m_1'}\ . \quad (5.14)$$

The expansion formula, of course, takes the form

$$f(a_1, a_2, a_3)$$

$$= \sum_{M_i} \int \mathrm{d}\lambda_i \int \mathrm{d}l_i \sum_{\tau_i, m_i} \sum_{j_i, m_i} F^{\lambda_i M_i}_{\tau_i l_i m_i, j_i m_i} \prod_{i=1,2,3} \mathcal{D}^{\lambda_i M_i}_{\tau_i l_i m_i', j_i m_i}(a_i). \quad (5.15)$$

We can derive a formula similar to (5.11) for $\mathcal{F}^{\lambda_i M_i}_{l_1 l_2 l_3 \tau_i j_i m_i}$ and also a Plancherel formula for f in terms of this coefficient; this completes the decomposition of the representation.

6. AN SL(2, C) EXPANSION AND A TOLLER-POLE MODEL

We have shown in sect. 4 that the general method of introducing analytic group variables, suitable for making hypotheses about the asymptotic behaviour of the six particle amplitude, results in its being defined as a function $\tilde{f}$ over three little groups moving in $\mathrm{Sl}(2, \mathbb{C})^3$. If we can extend this function to the whole of $\mathrm{SL}(2, \mathbb{C})^3$ then we can perform a partial-wave expansion in terms of the representation functions of SL(2, C), which will be well-behaved in the neighbourhood of $\lambda_{abc} = 0$. Of course, it will be necessary to incorporate the covariance conditions in the partial-wave coefficients (to make the Lorentz invariance of the function, when restricted to the little groups, apparent), particularly since this was the difficulty with the $\mathrm{SU}(1,1)^3$ expansion. This can be done more naturally if we can take $\tilde{f}$ to be a function on $\mathrm{SL}(2, \mathbb{C})^3$ satisfying covariance conditions with respect to subgroups of this group. The awkwardness of the changing nature of the covariance condition (4.10) can be avoided if we can arrange that $\tilde{f}$ is invariant over a larger subgroup of $\mathrm{SL}(2, \mathbb{C})^3$ in which this covariance group is always included. To maintain the maximum amount of freedom for the function restricted to the little groups, this larger covariance group must intersect each little group in exactly the appropriate original one-parameter subgroup. We can do this simply by taking advantage of the freedom available in choosing the moving standard triangle of the $\tilde{Q}^{0}$'s.

As in sect. 4, we choose the triangle formed by $\tilde{Q}^0_{\mathrm{A}}$, $\tilde{Q}^0_{\mathrm{B}}$, $\tilde{Q}^0_{\mathrm{C}}$ to lie in the plane containing the z-axis and the vector $(1+\lambda_{abc}, 0, 1-\lambda_{abc}, 0)$. But we also arrange that none of these lie along the z-axis at any stage. The covariance group $\hat{\mathrm{K}}$ defined by (4.10) will then always be contained in the little group of the z-axis : SU(1,1). Also this covariance group will, for any value of Q^2_{A}, Q^2_{B}, Q^2_{C}, be exactly the intersection of each of the little groups $\tilde{H}_{\mathrm{A}}$, $\tilde{H}_{\mathrm{B}}$, $\tilde{H}_{\mathrm{C}}$, with the little group of the z-axis. So if $\tilde{f}$ satisfies the covarianace condition (5.1), then its restriction to $\tilde{H}_{\mathrm{A}} \times \tilde{H}_{\mathrm{B}} \times \tilde{H}_{\mathrm{C}}$ will satisfy the correct covariance condition on the left.

The covariance conditions on the right (4.11) are not really a problem at $\lambda_{abc} = 0$ because they suffer no critical change and the covariance groups may be related to standard groups by isomorphisms depending analytically on the Q^2's. Consequently it is sufficient to ensure that $\tilde{f}$ satisfies

$$\tilde{f}(a_{\mathrm{A}}\hat{u}_{\mathrm{A}}, a_{\mathrm{B}}\hat{u}_{\mathrm{B}}, a_{\mathrm{C}}\hat{u}_{\mathrm{C}}) = \tilde{f}(a_{\mathrm{A}}, a_{\mathrm{B}}, a_{\mathrm{C}}) \qquad \hat{u}_{\mathrm{A}} \in \hat{\mathrm{U}}_{\mathrm{A}},\ \hat{u}_{\mathrm{B}} \in \hat{\mathrm{U}}_{\mathrm{B}},\ \hat{u}_{\mathrm{C}} \in \hat{\mathrm{U}}_{\mathrm{C}} \,. \tag{6.1}$$

It is not difficult to see that a function $\tilde{f}$ defined on $\tilde{H}_{\mathrm{A}} \times \tilde{H}_{\mathrm{B}} \times \tilde{H}_{\mathrm{C}}$ satisfying the covariance conditions (4.10) and (4.11) can be extended to one defined over $\mathrm{SL}(2, \mathbb{C})^3$ satisfying conditions (3.1) and (6.1).

The problem of performing an expansion in terms of representation functions of SL(2, C) for a function $\tilde{f}$ satisfying the non-compact covariance condition (5.1) was discussed in sect. 5. The covariance condition itself is equivalent to the restriction (5.14) on the partial wave coefficients. This is a restriction on the left hand labels $\tau_i l_i m_i$ of the partial wave coefficients $\tilde{F}^{\lambda_i M_i}_{\tau_i l_i m_i', j_i m_i}$. (6.1) will be equivalent to restrictions on the right hand labels of the form

$$\tilde{F}^{\lambda_i M_i}_{\tau_i l_i m_i', j_i m_i} = \sum_{j_i'} \mathcal{G}^{\lambda_i M_i}_{\tau_i l_i m_i', j_i'} \mathcal{D}^{\lambda_i M_i}_{j_i m_i, j_i' 0} (\tilde{g}_i^{-1}) , \tag{6.2}$$

still taking $\tilde{d}_A$ etc., to be the identity.

Thus, with these restrictions on the partial-wave coefficients, we can perform the expansion (5.15). As usual [3, 20] we can expect to write this expansion in terms of second-type functions, and draw the contours to the left to reveal pole contributions in the λ planes (assuming the partial wave coefficients to be meromorphic in some strip). This will give an asymptotic expansion of the function in appropriate limits in $SL(2, \mathbb{C})^3$ including those in which the 'adjusted' size of the variables $\tilde{h}$ becomes larger in the sense we defined at the end of sect. 4.

So if we now consider a single triple-pole contribution

$$\sum_{\substack{\tau_i, l_i, m_i' \\ j_i m_i}} \beta_{\tau_i l_i m_i', j_i m_i}(Q_A^2, Q_B^2, Q_C^2) \, \mathcal{A}^{\alpha_i(Q_i^2), M_i}_{\tau_i l_i m_i', j_i m_i}(\tilde{h}_i) , \tag{6.3}$$

occurring at $\lambda_i = \alpha_i(Q_i^2)$, with a definite value of M_i and a factorized residue

$$\beta_{\tau_i l_i m_i', j_i m_i}(Q_A^2, Q_B^2, Q_C^2)$$

$$= \beta^{(v)}_{\tau_i l_i m_i'}(Q_A^2, Q_B^2, Q_C^2)\, \beta^{(A)}_{j_1 m_1}(Q_A^2) \beta^{(B)}_{j_2 m_2}(Q_B^2) \beta^{(C)}_{j_3 m_3}(Q_C^2) ,$$

we can use the general results of Sciarrino and Toller [14] on the asymptotic relation of second-type Lorentz-group functions to second type SU(1,1) functions to infer that (6.3) corresponds to an infinite sequence of triple Regge-pole contributions with factorized residues. Toller's results relate the second-type $SL(2, \mathbb{C})$ functions to the second-type SU(1,1) functions when restricted to the standard SU(1,1) subgroup. To use these results it is necessary to employ $\tilde{g}_A$, etc., to map $\tilde{H}_A$ onto this SU(1,1) subgroup and write

$$\mathcal{D}^{\lambda M}_{jmj'm'}(\tilde{h}_A)$$

$$= \sum_{j_1, m_1, j_2, m_2} \mathcal{D}^{\lambda M}_{jmj_1 m_1}(\tilde{g}_A^{-1}) \, \mathcal{D}^{\lambda M}_{j_1 m_1 j_2 m_2}(\tilde{g}_A \tilde{h}_A \tilde{g}_A^{-1}) \, \mathcal{D}^{\lambda M}_{j_2 m_2 j' m'}(\tilde{g}_A) , \tag{6.4}$$

in the expansion formula before changing to second-type functions of argument $\tilde{g}_A \tilde{h}_A \tilde{g}_A^{-1}$. This gives, instead of (6.3) the contribution

$$\sum_{\substack{\tau_i,\, l_i,\, m_i' \\ j_i,\, m_i}} \beta'_{\tau_i l_i m_i', j_i m_i}(Q_A^2, Q_B^2, Q_C^2) \prod_{X=1,2,3} \sum_{j_1', m_1'} \mathcal{D}^{\lambda M}_{\tau_X l_X m_X', j_1 m_1}(\tilde{g}_X^{-1}) \mathcal{A}^{\alpha_X(Q_X^2),\, M_X}_{j_1 m_1, j_X m_X}(h_X) , \qquad (6.5)$$

where we have used (6.2).

This expression together with the asymptotic expansion of $\mathcal{A}^{\alpha_X(Q_X^2), M_X}_{j_1 m'_1, j_2 m_2}(h_X)$ as a series of second-type functions of SU(1.1) shows that, both in the s-t and s-s regions and on the boundary $\lambda_{abc} = 0$, a triple Toller corresponds to infinite sequences of triple Regge poles with parallel trajectories.

Because the second-type function in (6.3) only has singularities for bounded values of $\tilde{h}$ it is a possible uniform asymptotic approximation to the amplitude in a neighbourhood of $\lambda_{abc} = 0$.

7. THE THREE-REGGEON VERTEX AND THE n-PARTICLE AMPLITUDE

The aim of this section is to describe and extend the work of Cosenza, Sciarrino and Toller [3-6] and of Bali, Chew and Pignotti [2] on the multi-Reggeon description of multiparticle amplitudes. Using an approach very similar to that of Toller, and relying heavily on his theorems [5] on the incorporation of analyticity, we discuss the role of the three-Reggeon vertex in a general group theoretic description of multiparticle amplitudes.

Following Cosenza, Sciarrino and Toller, we consider a process involving n particle with arbitrary spins and masses. The connected part of the scattering amplitude for this process can be described by a function $\hat{M}_{m_1,\ldots,m_n}(P^{(1)},\ldots,P^{(n)})$ of the spin labels m_i and of the momenta $P^{(i)}$ satisfying the mass shell constraints $P^{(i)2} = M_i^2$ and momentum conservation $\sum_i P^{(i)} = 0$. As before, the sign of the energy component of $P^{(i)}$ will be positive (negative) if the ith particle is outgoing (incoming). Now, to study analyticity, we consider complex values of the $P^{(i)}$, and assume that $\hat{M}$ is analytic apart from certain singularities such as those required by unitarity. (Note that, the assumption of crossing made here, that is, that the various channels are described by just one analytic function of the $P^{(i)}$, is not essential for the following discussion). For the group theoretic treatment of particles with spin it is convenient to introduce another function M of complex Lorentz transformations a_i (or more precisely elements of $SL(2, \mathbb{C})^c$, the universal covering group of the complex Lorentz group), related to the $\hat{M}$ function by

$$M_{m_1,\ldots,m_n}(a_1,\ldots,a_n) = \sum_{m_i'} \Lambda^{(1)}_{m_1 m_1'}(a_1^{-1}) \ldots\ldots \Lambda^{(n)}_{m_n m_n'}(a_n^{-1}) \times \hat{M}_{m_1',\ldots,m_n'}(L(a_1)p(1)^0,\ldots, L(a_n)p(n)^0) , \qquad (7.1)$$

and defined over the subset of $[\mathrm{SL}(2, \mathbb{C})^c]^n$ satisfying

$$\sum_i L(a_i)p(i)^{\mathrm{o}} = 0 , \qquad \text{where} \qquad p(i)^{\mathrm{o}} = (M_i, 0, 0, 0) \tag{7.2}$$

where $\Lambda^{(i)}$ denotes the appropriate analytic representation of $\mathrm{SL}(2, \mathbb{C})^c$. Toller [5] has made precise and proved the statement that M is free of kinematic singularities and constraints and so will have exactly the dynamical singularities that $\hat{M}$ has. M satisfies the invariance condition

$$M_{m_1,\ldots,m_n}(aa_1,\ldots,aa_n) = M_{m_1,\ldots,m_n}(a_1,\ldots,a_n) , \quad a \in \mathrm{SL}(2,\mathrm{C})^c , \tag{7.3}$$

which for $a = t^{(1)}$ is a statement of TCP invariance; and also n covariance conditions of the form

$$M_{m_1, m_2,\ldots,m_n}(a_1h_1, a_2,\ldots a_n)$$

$$= \sum_{m_i'} R^{j_1}_{m_1'm_1}(h_1)M_{m_1', m_2,\ldots,m_n}(a_1, a_2,\ldots,a_n) , \quad h_1 \in H_+^{\mathrm{C}} , \tag{7.4}$$

Consider a simply connected graph or tree diagram with n external lines, r internal lines, each joining two of the $(r+1)$ vertices (but with no set of internal lines forming a closed loop). Associating each of the n external lines with one of the n particles involved in the process, for given external momenta $P^{(i)}$ a (unique) momentum Q_i is associated with each internal line so that momentum is conserved at each vertex. The next stage is to define the amplitude as a function of elements of the little groups of standard vector associated with each of the internal lines and, essentially, the Lorentz invariants which can be formed from the momenta meeting at each vertex. In general, an arbitrary number of external and internal lines could meet at a vertex. If a vertex has more than three lines we can divide these into two sets, separating the vertex into two new vertices joined by a new internal line (as indicated in fig. 2).

Thus we obtain a new tree diagram from which the original diagram may be obtained by contraction of an internal line. The invariants which could be formed from the momenta meeting at the old vertex, and fix their relative position, are equivalent to the two sets of invariants formed from the momenta meeting at each of the two new vertices, respectively, together with the little group element corresponding to the new internal line (which determines the relative position of the momenta at one new vertex relative to those at the other). If this process is applied repeatedly until no more than three lines meet at any vertex, we obtain a 'fully-extended' diagram, in which each vertex has either three internal lines ('three-Reggeon' vertex) or two internal lines and one external line ('two-Reggeon particle' vertex) or one internal line and two external lines, ('Reggeon two-particle' vertex).

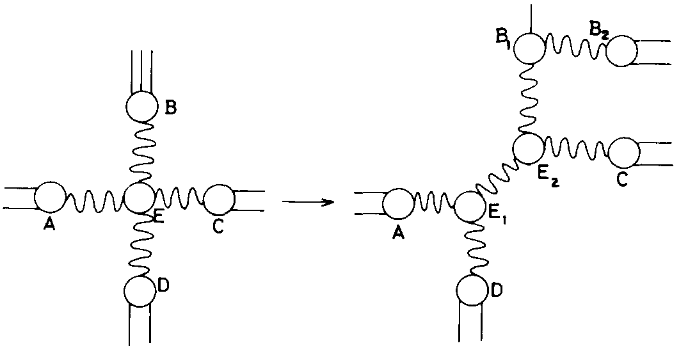

Fig. 2. Extension of tree diagram into a fully-extended diagram.

Any tree diagram can be obtained by contraction of internal lines from a 'fully-extended' diagram and the Regge limits are a subset of those that can be obtained from the larger diagram. Thus, it is sufficient to discuss in detail the definition of the amplitude as a function over the little groups corresponding to a 'fully-extended' diagram *.

Three lines will meet at a vertex in such a diagram, we will denote the momenta associated with them by Q_i, Q_j and Q_k (irrespective of whether they are internal or external lines), and for convenience suppose that the momenta have been directed so that momentum conservation takes the form $Q_i + Q_j + Q_k = 0$ for this vertex. We choose a standard triangle for the vertex consisting of vectors $Q_i^{\rm o}$, $Q_j^{\rm o}$, $Q_k^{\rm o}$ depending on Q_i^2, Q_j^2 and Q_k^2 such that $Q_i^{{\rm o}2} = Q_i^2$, $Q_j^{{\rm o}2} = Q_j^2$, $Q_k^{{\rm o}2} = Q_k^2$. To introduce analyticity this dependence should be analytic in a neighbourhood of the values of Q_i^2, Q_j^2, Q_k^2 considered. If one of the lines corresponds to an external particle of non-zero mass, we may clearly always take the standard triangle to lie in the z, t plane. As we have seen, if the vertex consists of three internal lines the situation is somewhat different; if all three Q's are spacelike then we have to arrange an analytic transition from the s-s to the s-t regions. In general we can choose an element $c \in {\rm SL}(2, {\bf C})$, for the vertex which takes the standard triangle $\{Q_i^{\rm o}, Q_j^{\rm o}, Q_k^{\rm o}\}$ into the actual triangle $\{Q_i, Q_j, Q_k\}$ so that

$$Q_i = L(c)Q_i^{\rm o}\,, \qquad Q_j = L(c)Q_j^{\rm o}\,, \qquad Q_k = L(c)Q_k^{\rm o}\,. \tag{7.5}$$

If i is an internal line it will run from the vertex, v_1 say, to the vertex v_2, and will have standard vectors $Q_{i(1)}^{\rm o}$ and $Q_{i(2)}^{\rm o}$ in the respective standard triangles of these vertices. If c_1 and c_2 denote the transformations

* The relation of fully extended diagrams to coupling schemes for the products of the Hilbert spaces of the external particles has been discussed by Toller in ref. [11].

corresponding to these two vertices defined by equations (7.5) then

$$L\,(c_2^{-1}\,c_1)\,Q^{\rm o}_{i(1)} = Q^{\rm o}_{i(2)}\;. \tag{7.6}$$

Consequently, if we introduce a transformation q_{12} which has the property

$$L\,(q_{12})\,Q^{\rm o}_{i(1)} = Q^{\rm o}_{i(2)}\;, \tag{7.7}$$

and depends on the squares of the five vectors at the two vertices, we have that $q_{12}^{-1}c_2^{-1}\,c_1$, is an element of the little group of $Q^{\rm o}_{i(1)}$ and $c_2^{-1}c_1\;q_{12}^{-1}$ is an element of the little group of $Q^{\rm o}_{i(2)}$. Thus in this way, for given external momenta, we can associate with each internal line elements of the little group of either of the standard vectors associated with it.

In fact Toller [5] has shown that the c_i defined by the equation (7.5) can be chosen to be analytic functions of the external momenta if the standard triangle have been chosen in a suitable analytic way. It also follows from results of his that the q_{ij} we have introduced in (7.7) may be taken to be analytic. In each case we have to exclude a certain subset of the external momenta but this is so small that it is irrelevant [5].

Instead of using either of the standard vectors $Q^{\rm o}_{i(1)}$ or $Q^{\rm o}_{i(2)}$ from the the standard vertex triangles, an alternative procedure is to introduce a third analytically varying standard vector $Q_i^{\rm o}{}'$, linked to $Q^{\rm o}_{i(j)}$ by analytically varying transformations, and construct a little group element of $Q_i^{\rm o}{}'$ from $c_2^{-1}c_1$. This is the procedure adopted by Toller in his recent paper [6].

An obvious modification of our formalism, which will simplify it from certain points of view, is to match up the standard triangles at adjacent vertices so that, for example, the new $Q^{\rm o}_{i(2)} = Q^{\rm o}_{i(1)}$. We can do this by working outwards from a given vertex, each time applying $L(q_{ij})$ to the vertices further out. (The standard triangles will now, in general, depend on all the Q_i^2's). It is then unnecessary to introduce new q_{ij}'s and $c_2^{-1}c_1$ will be an element of the little group of the new $Q_i^{\rm o}$.

The action of the little group elements can now be pictured as follows. If we consider an arbitrary fully-extended diagram (for example fig. 3(a)) we can construct a dual diagram from the momenta P_i, Q_j which takes the form of a network of triangles (see fig. 3(b)), one triangle for each vertex. The little group elements $h_i = c_2^{-1}c_1$ etc., determine the relative position of adjacent triangles in the network. The standard configurations of the vertex triangles now fit together to form an analytically varying standard position of this network and the measurement of the relative positions is based on this standard network.

If all the external particles are spinless, all we are interested in doing is determining the external momenta P_i to within an overall Lorentz

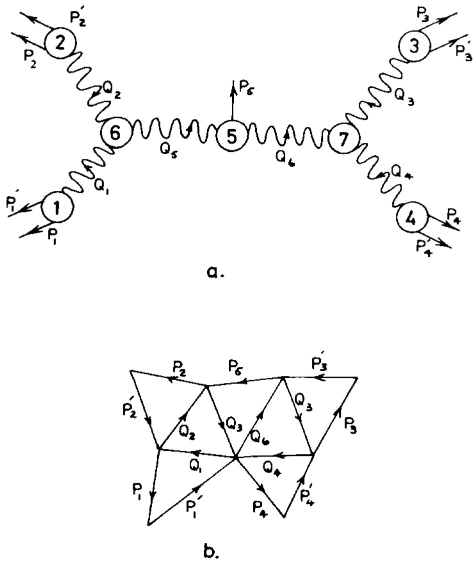

Fig. 3. (a) A fully-extended diagram. (b) The dual diagram corresponding to (a).

transformation. It is easy to do this in terms of the little group elements (and the Q^2's). For if v_0 is any fixed vertex, $L(c_0^{-1})P_i$ is the product of little group elements corresponding to the internal lines forming the unique path from v_0 to the external vertex at which P_i is attached. For example in fig. 3 if we take $v_0 \equiv v_5$

$$\begin{aligned} L(c_5^{-1})P_1 &= L(c_5^{-1}c_1)P_1^0 \\ &= L(c_5^{-1}c_6c_6^{-1}c_1)P_1^0 \\ &= L(h_5^{-1}h_1)P_1^0 \ , \end{aligned} \tag{7.8}$$

where h_i is an element of H_i the little group of Q_i^0 and P_i^0 is the vector in the standard triangle of the vertex v_1 corresponding to P_1, and is an analytic function of the external masses and Q_i^2 $(1 \leq i \leq 6)$ in general. Thus if we define a function f by

$$f(h_1, h_2, h_3, h_4, h_5, h_6) = \hat{M}(L(h_5^{-1}h_1)P_1^0 \, , \, L(h_5^{-1}h_2)P_2^0 \, , \\ L(h_6^{-1}h_3)P_3^0 \, , \, L(h_6^{-1}h_4)P_4^0 \, , \, P_5^0), \quad h_i \in H_i \tag{7.9}$$

where now P_i^0 stands for P_i^0 and $P_i^{0\prime}$ $(1 \leq i \leq 4)$, it will be analytic when

$\hat{M}$ is and we can reconstruct the $\hat{M}$ function from it, because

$$\hat{M}(P_1, P_2, P_3, P_4, P_5)$$

$$= \hat{M}(L(c_5^{-1})P_1, L(c_5^{-1})P_2, L(c_5^{-1})P_3, L(c_5^{-1})P_4, L(c_5^{-1})P_5)$$

$$= \hat{M}(L(h_5^{-1}h_1)P_1^{\mathrm{o}}, L(h_5^{-1}h_2)P_2^{\mathrm{o}}, L(h_6^{-1}h_3)P_3^{\mathrm{o}}, L(h_6^{-1}h_4)P_4^{\mathrm{o}}, P_5^{\mathrm{o}}) \,. \quad (7.10)$$

Therefore f will have no kinematic singularities or constraints by Toller's theorems [5]. This procedure which can obviously be applied to any fully extended diagram is a direct generalisation of our previous treatment of the spinless six particle amplitude and is obviously independent of the choice of the special vertex v_{o}. Only small amendments to this procedure are now necessary to introduce external particles with spin. If the M function is used instead of the $\hat{M}$ function to define the function f over the little groups then the covariance conditions for the spin labels (7.4) will give simple covariance conditions for f. Thus we must relate the transformations a_i associated with each of the external lines, to the c-transformations and corresponding standard triangles introduced at each of the external vertices. We have from (7.1), (7.2) and (7.5) that, for the i-th external particle

$$\begin{aligned} P_i &= L(a_i)p_{(i)}^{\mathrm{o}} \\ &= L(c_i)P_i^{\mathrm{o}} \\ &= L(c_i b_i)p_{(i)}^{\mathrm{o}} \,, \end{aligned} \quad (7.11)$$

where c_i is the c-transformation for the vertex to which P_i is attached, P_i^{o} is the relevant standard vector and b_i is an analytic transformation relating P_i^{o} and $p_{(i)}^{\mathrm{o}}$. Therefore

$$a_i = c_i b_i u_i \,, \qquad u_i \in \mathrm{H}_+^{\mathrm{C}} \,. \quad (7.12)$$

Since c_i and b_i have been chosen to be analytic functions of the external momenta and therefore of the a_i's, u_i will also be an analytic function of the a_i's. Therefore, if we define $f(h_1, \ldots, h_6)$ for the diagram of fig. 3 by

$$f_{m_1, m_1', \ldots\, m_4, m_4', m_5}(h_1, \ldots, h_6)$$

$$= M_{m_1, m_1', \ldots\, m_4, m_4', m_5}(h_5^{-1}h_1 b_1, h_5^{-1}h_1 b_1', \ldots, h_6^{-1}h_4 b_4, h_6^{-1}h_4 b_4', b_5) \,,$$

$$h_i \in \mathrm{H}_i \,. \quad (7.13)$$

it is clear that the M-function can be reconstructed from f using (7.3), (7.4) and (7.12) and so again by Toller's theorems f will have no kinematic singularities or constraints. This procedure is exactly analogous to that followed for the spinless amplitude and can also be applied to any fully extended diagram.

From (7.13) f will satisfy certain covariance conditions and these ensure that the reconstruction of the M-function is well defined. If we consider an internal (or three-Reggeon) vertex (for example vertex 6 of fig. 3) with momenta Q_i, Q_j, Q_k meeting, then the function f will be invariant under the transformation

$$h_i \to kh_i \,, \quad h_j \to kh_j \,, \quad h_k \to kh_k \,, \quad k \;\; \mathrm{H}_i \cap \mathrm{H}_j \cap \mathrm{H}_k = \mathrm{K} \;. \tag{7.14}$$

K will be the group of transformations which leaves invariant the plane of the standard triangle at the vertex. This covariance condition exactly corresponds to the possible freedom in the choice of a c satisfying (7.5). To make the covariance conditions at the external vertices simple, it is convenient to require that in addition to satisfying $L(b_i)p^{\mathrm{o}}_{(i)} = P^{\mathrm{o}}_i$, b_i should also map the (z,t) plane into the plane of the standard triangle at the corresponding vertex. The covariance condition at a two-Reggeon particle vertex (for example vertex 5 of fig. 3), where P_i, Q_j, Q_k meet, takes the form that under the transformation

$$h_j \to h_j u_i \;\; h_k \to h_k u_i \,, \quad \text{where } u_i = b_i\, u_z(\mu) b_i^{-1} \in \mathrm{H}_i \cap \mathrm{H}_j \cap \mathrm{H}_k = \mathrm{U}_i \,, \tag{7.15}$$

f is multiplied by the phase factor $\mathrm{e}^{-im_i\mu}$. For a Reggeon two-particle vertex (for example vertex 1 in fig. 3), where P_i, P_j, Q_k meet, the transformation

$$h_k \to h_k u_{ij} \,, \quad \text{where } u_{ij} = b_i u_z(\mu) b_i^{-1} = b_j\, u_z(\mu) b_j^{-1} \in \mathrm{H}_i \cap \mathrm{H}_j \cap \mathrm{H}_k = \mathrm{U}_{ij} \,, \tag{7.16}$$

produces a phase factor $\mathrm{e}^{-i(m_i+m_j)\mu}$ on f. Both u_i and u_{ij} preserve the planes of their respective standard triangles.

Having defined the amplitude as function over the little groups H_i we can now perform a partial wave analysis by expanding in terms of repressentation functions of these groups. To construct these functions it is necessary to parametrize the groups by defining an isomorphism of the moving groups on to the standard SU(1,1) little group of the z-axis in the region $Q_i^2 < 0$ in which we are interested, as in sect. 4. Such an isomorphism will be singular at $Q_i^2 = 0$. This isomorphism will be of the form $h_i \to g_i h_i g_i^{-1} : \mathrm{H}_i \to \mathrm{SU}(1,1)$, where $g_i \in \mathrm{SL}(2, \mathbb{C})$. In order to conveniently incorporate the covariance conditions we introduce elements of SU(1.1) which map the image of the plane of the standard triangles under g_i on to either the (y,z) or (z,t) plane as is appropriate. For the external vertices these take the form of transformations d_i which will be analytic (as a function of the Q^2's). For three-Reggeon vertices we introduce transformations r_i which, as we showed in sect. 4, will necessarily be singular at $\lambda(Q_i^2, Q_j^2, Q_k^2) = 0$ and are different for $\lambda_{ijk} \gtrless 0$. To do the expansion at $\lambda_{ijk} = 0$ a different r-transformation which maps on to a standard plane touching the light cone must be used.

We can now write the general form of an expansion as follows: We divide the internal lines i into four sets $\alpha, \beta, \gamma, \delta$, depending on whether they have three-Reggeon vertices on both sides, just on the right, just on the left, or on neither side. Then

$$f_{m_1,\ldots,m_n}(h_1,\ldots,h_r) = \sum_{n_i,n_i'} \int \mathrm{d}\Lambda_i \left\{ \prod_{i\in\alpha} D^{\Lambda_i}_{n_i n_i'}(r_i g_i h_i g_i^{-1} r_i') \prod_{i\in\beta} D^{\Lambda_i}_{n_i n_i'}(d_i g_i h_i g_i^{-1} r_i') \prod_{i\in\gamma} D^{\Lambda_i}_{n_i n'}(r_i g_i h_i g_i^{-1} d_i') \prod_{i\in\delta} D^{\Lambda_i}_{n_i n'}(d_i g_i h_i g_i^{-1} d_i') \right\} F^{\Lambda_1\ldots\Lambda_r,\, m_1\ldots m_n}_{n_1 n_1'\ldots n_r n'} \tag{7.17}$$

Here Λ_i labels the representations of SU(1,1), D^{Λ_i} is a representation function and n_i, n_i' denote discrete labels unless the left, or right end respectively, of the line i is part of three-Reggeon vertex in the s-s region (or on the boundary of the s-s and s-t regions) when it is continuous. The partial wave coefficient F will be zero unless the sum of the m and n labels for each vertex is zero.

The difficulties that occur when either a g_i or an r_i become singular and their relative importance have been discussed in sect. 3 and 4. They can be removed by extending the function f to a function on $SL(2, \mathbb{C})^r$, satisfying the appropriate covariance conditions, and performing a partial - wave expansion in terms of the representation functions of SL(2, $\mathbb{C}$). The covariance conditions at the external vertices present no problem (if no two external particles have the same mass). They can be incorporated by introducing a d_i transformation as for the SU(1,1) expansion and again making the partial-wave coefficient

$$F^{\Lambda_i\ldots\Lambda_r\, m_3,\ldots m_n}_{n_1 j_1, n_1' j_1', \ldots n_r j_r, n_r' j_r'}$$

zero unless the sum of the m and n labels for each vertex is zero. To incorporate the covariance condition at a three-Reggeon vertex, we introduce transformations s_i which (analytic as a function of the Q_i's) align the plane of the standard triangle to include the z-axis, but with none of the sides along the z-axis. The covariance condition will then be satisfied if the j, n labels corresponding to this vertex are taken to be the SU(1,1) basis labels introduced in sect. 5, and F satisfies the condition (5.14). With these restrictions on F the expansion will take the same form as that given in (7.17) expect that the g_i's will not be present, the r_i's will be replaced by s_i's, the Λ_i's will denote representations of SL(2, $\mathbb{C}$) and the D^{Λ_i}'s will be the corresponding representation functions. Toller poles can now be introduced into this expansion giving the usual sequences of Regge poles.

The difficulties at pseudothresholds of the external particles, which occur in our formalism as a result of the d_i's becoming singular have been treated in detail by Cosenza, Sciarrino and Toller [4]. When the masses of the external particles are not equal these occur for values of the corresponding Q_i^2, greater than zero. They have also considered the more complicated problem where two external particles of equal mass meet at one vertex and the corresponding d_i and g_i become singular simultaneously.

We are grateful to Dr. D. I. Olive for his interest and encouragement during this work. We are also grateful to the SRC for maintenance grants.

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